THE ISOMORPHISM THEOREM FOR RELATIVELY FINITELY DETERMINED $\mathbf{Z}^n$-ACTIONS

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ABSTRACT
Any two ergodic $\mathbf{Z}^n$-actions which are finitely determined relative to a common factor are isomorphic if and only if they have the same entropy.

The Isomorphism Theorem for finitely determined Z-actions is a well known result first proven by D. Ornstein [7]. In [1], R. Burton and A. Rothstein prove this same theorem by applying Baire Category techniques to spaces of joinings of finitely determined processes. The Isomorphism Theorem for relatively finitely determined Z-actions was set forth by J. P. Thouvenot [9].

A proof of the Isomorphism Theorem for relatively finitely determined $\mathbf{Z}^n$-actions is presented, utilizing the techniques developed by Burton and Rothstein.

Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the n fundamental basis vectors of $\mathbf{Z}^n$. Let $\{T_{\mathbf{e}_1}, \ldots, T_{\mathbf{e}_n}\}$ be n commuting measure preserving transformations on X. Let $\mathcal{T} = \{T_{\mathbf{v}}\}_{\mathbf{v}\in\mathbf{Z}^n}$ be the group of transformations generated by $\{T_{\mathbf{e}_1}, \ldots, T_{\mathbf{e}_n}\}$, $T_{\mathbf{v}}(x) = T_{\mathbf{e}_1}^{v_1} \circ \cdots \circ T_{\mathbf{e}_n}^{v_n}(x)$, $\mathbf{v} = (v_1, \ldots, v_n)$.

The system $(X, \mathcal{A}, \mu), \mathcal{T}$ is ergodic if the only functions f which satisfy $f(x) = f(T_{\mathbf{v}}x)$ for all $\mathbf{v} \in \mathbf{Z}^n$ are constant functions.

Let $R(m) \subset \mathbf{Z}^n$ be the cube of size $(2m+1)^n$ centered at the origin. Let P be a finite partition of X. Let $\mathcal{P}_m = \bigvee_{\mathbf{v}\in R(m)} T_{-\mathbf{v}}(P)$. Define the entropy of the

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process $(\mathcal{T}, P)$ by $h(\mathcal{T}, P) = \lim_{m\to\infty} h(\mathcal{P}_m)(2m+1)^{-n}$ where $h(\mathcal{P}_m)$ is the usual entropy of a finite partition. J. P. Conze presents a complete discussion of the notion of entropy for $\mathbf{Z}^n$-actions in [2].

An ergodic $\mathbf{Z}^n$-process $(\mathcal{T}, P \vee H)$ is called *H-relatively finitely determined* (written H-rel. F.D.) if for every $\varepsilon > 0$ there exists $\delta > 0$ and M such that for all ergodic processes $(\mathcal{T}', P' \vee H')$ with $(\mathcal{T}', H') \sim (\mathcal{T}, H)$, the conditions

(i) $|h(\mathcal{T}, P \vee H) - h(\mathcal{T}', P' \vee H')| < \delta$ and

(ii) $$\left|\operatorname{dist}\left(\bigvee_{\mathbf{v}\in R(M)} T_{\mathbf{v}}(P \vee H)\right) - \operatorname{dist}\left(\bigvee_{\mathbf{v}\in R(M)} T'_{\mathbf{v}}(P' \vee H')\right)\right| < \delta,$$

imply that $\bar{d}_{H,H'}[(\mathcal{T}, P \vee H), (\mathcal{T}', P' \vee H')] < \varepsilon$.

The notation $(\mathcal{T}', H') \sim (\mathcal{T}, H)$ means that $\bigvee_{\mathbf{v}\in R(m)} T_{\mathbf{v}}(H) = \bigvee_{\mathbf{v}\in R(m)} T'_{\mathbf{v}}(H')$ for all $m \geqq 0$. The relative $\bar{d}$-metric, first defined by J. P. Thouvenot in [9] and also described by A. Fieldsteel in [3], is just the regular $\bar{d}$-metric, in which we restrict ourselves to joinings which restrict to the diagonal joining on the common factor $(\mathcal{T}', H') \sim (\mathcal{T}, H)$.

We say that $\mathcal{T}$ *is isomorphic to* $\mathcal{T}'$ *relative to the common factor* $(\mathcal{T}', H') \sim (\mathcal{T}, H)$ [written $\mathcal{T} \underset{H,H'}{\cong} \mathcal{T}'$] if there is a measurable isomorphism $\phi : (X, \mathcal{F}, m) \to (X', \mathcal{F}', m')$ such that $\phi(T_{\mathbf{v}}x) = T'_{\mathbf{v}}\phi(x)$ for all $\mathbf{v} \in \mathbf{Z}^n$ and a.e. $x \in X$, and such that $\phi(H) = H'$.

ISOMORPHISM THEOREM FOR RELATIVELY-F.D. $\mathbf{Z}^n$-ACTIONS. *For $i = 1, 2$ let $(X_i, \mathcal{F}_i, m_i)$, $\mathcal{T}^i$ be two ergodic $\mathbf{Z}^n$-actions such that $\mathcal{F}_i = \bigvee_{\mathbf{v}\in\mathbf{Z}^n} T^i_{-\mathbf{v}}(P_i \vee H_i)$, where P_i and H_i are finite partitions of X_i, and such that $(\mathcal{T}^i, P_i \vee H_i)$ is H_i-rel. f.d. Suppose $(\mathcal{T}^1, H_1) \sim (\mathcal{T}^2, H_2)$. Then $h(m_1 : \mathcal{T}^1) = h(m_2 : \mathcal{T}^2)$ if and only if $\mathcal{T}^1 \underset{H_1,H_2}{\cong} \mathcal{T}^2$.*

Form the product space $(Z, \mathcal{F}_z)$, $\mathcal{U} = (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2)$, $\{T^1_{\mathbf{v}} \times T^2_{\mathbf{w}}\}_{(\mathbf{v},\mathbf{w})\in\mathbf{Z}^{2n}}$, a $\mathbf{Z}^{2n}$-action. Following Burton and Rothstein, we let 2 denote the trivial algebra whenever necessary.

Define a space of joinings on Z by $\mathcal{M}_{H_1,H_2}(Z) = \{\mu \mid \mu$ is a $\mathcal{U}$-invariant ergodic probability measure on Z such that $\mu^1 = m_1$, $\mu^2 = m_2$ and such that $\bigvee_{\mathbf{v}\in\mathbf{Z}^n} T^1_{-\mathbf{v}}(H_1) \times 2 = 2 \times \bigvee_{\mathbf{v}\in\mathbf{Z}^n} T^2_{-\mathbf{v}}(H_2)$ μ-a.s.$\}$. Note μ^i denotes the ith marginal of μ. We call $\mathcal{M}_{H_1,H_2}(Z)$ the space of ergodic joinings of $(X_1, \mathcal{F}_1, m_1)$, $\mathcal{T}^1$ and $(X_2, \mathcal{F}_2, m_2)$, $\mathcal{T}^2$ relative to the common factor $(\mathcal{T}^1, H_1) \sim (\mathcal{T}^2, H_2)$. For ease of notation write $\mathcal{M} = \mathcal{M}_{H_1,H_2}(Z)$.

Let $\{C_i\}$ be an enumeration of the cylinder sets of $(\mathcal{T}^1, P_1 \vee H_1)$ and let $\{D_j\}$

be an enumeration of the cylinder sets of $(\mathcal{T}^2, P_2 \vee H_2)$. Define a metric on $\mathcal{M}$ by

$$d(\mu, \nu) = \sum_{i,j} \frac{|\mu(C_i \times D_j) - \nu(C_i \times D_j)|}{2^{i+j}}.$$

Call this the distribution metric on relative joinings.

Let $\mathcal{M}_0$ be the closure of $\mathcal{M}$ with respect to this distribution metric. The space $\mathcal{M}$ is nonempty. Specifically, if ν is the relatively independent joining of m_1 and m_2 over the factor $(\mathcal{T}^1, H_1) \sim (\mathcal{T}^2, H_2)$ then ν satisfies the requirements of $\mathcal{M}$, except possibly ergodicity. It is straightforward to show that since $(X_1, \mathcal{T}^1)$ and $(X_2, \mathcal{T}^2)$ are ergodic, ν-a.e. ergodic component ν_z of ν is itself a joining of m_1 and m_2 which lies in $\mathcal{M}$.

It is easy to show that, with respect to the distribution metric, $\mathcal{M}_0$ is a compact metric space. Furthermore, $\mathcal{M}$ is a dense G_δ in $\mathcal{M}_0$.

Let

$$\mathcal{M}_1 = \{\mu \in \mathcal{M}_0 \mid h_\mu(P_1 \vee H_1 \times 2 \mid 2 \times \mathcal{F}_2) = 0\}$$

and let

$$\mathcal{M}(Z) = \{\mu \in \mathcal{M}_0 \mid h_\mu(2 \times P_2 \vee H_2 \mid \mathcal{F}_1 \times 2) = 0\}.$$

The proof of the relative isomorphism theorem reduces to proving the following theorem.

THEOREM 1. *$\mathcal{M}_1$ is a dense G_δ in $\mathcal{M}_0$.*

To see that this completes the isomorphism theorem, suppose, for a moment, that Theorem 1 is true. A symmetric argument implies that $\mathcal{M}_2$ is also a dense G_δ in $\mathcal{M}_0$, and hence $J = \mathcal{M} \cap \mathcal{M}_1 \cap \mathcal{M}_2$ is a dense G_δ in $\mathcal{M}_0$. Thus J is the set of $\mathcal{U}$-invariant ergodic joinings of m_1 and m_2, which respect the common factor $(\mathcal{T}^1, H_1)$, such that for every $\mu \in J$, $h_\mu((P_1 \vee H_1) \times 2 \mid 2 \times \mathcal{F}_2) = 0$ and $h_\mu(2 \times (P_2 \vee H_2) \mid \mathcal{F}_1 \times 2) = 0$. Any such joining gives rise to an isomorphism of $\mathcal{T}^1$ and $\mathcal{T}^2$, relative to the common factor $(\mathcal{T}^1, H_1)$. Specifically, the existence of such a joining implies that there is a measurable, measure preserving isomorphism between the σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$, relative to the common factor. This isomorphism induces a pointwise isomorphism of $\mathcal{T}^1$ and $\mathcal{T}^2$, relative to the common factor.

It remains to prove Theorem 1. We need the following two technical lemmas. The first describes a method of slightly boosting entropy. The second lemma describes a method of copying a process "into" another of strictly lower entropy while preserving a common factor. These two lemmas are adaptations of work by Burton and Rothstein [1].

LEMMA 2. *Let* $(X, \mathcal{F}, P), \mathcal{T}$ *be a* $\mathbf{Z}^n$*-sequence space on* k *symbols* $\{1, 2, \ldots, k\}$. *Let* $(E, \mathcal{E}, B \times P), \mathcal{S}$ *be the product space,* $E = X \times X$, $\mathcal{E} = \mathcal{F} \times \mathcal{F}$, $\mathcal{S} = \mathcal{T} \times \mathcal{T}$. *Let* m *be a* $\mathcal{T}$*-invariant ergodic probability measure on* $\mathcal{F}$. *Let* $t \in [0, 1]$.

There exists an $\mathcal{S}$*-invariant measure* ζ *on* $\mathcal{E}$ *such that* $\zeta^2 = m$ *and* $h(\zeta^1 : \mathcal{T}, P) \geqq (1 - t)h(m : \mathcal{T}, P) + t(\log k)$. *Furthermore,* $\zeta(\bigcup_{i=1}^{k} P_i \times P_i) \geqq 1 - t$.

Burton and Rothstein call ζ the *t-randomization of m*. They prove this result, and a bit more, about this randomization in [1]. Their proof carries over completely to the $\mathbf{Z}^n$-action case, and thus will not be repeated here. Instead, we simply outline the construction of the measure ζ.

Let m' be a measure so that $(X, \mathcal{F}, P, m'), \mathcal{T}$ is the Bernoulli shift with $m'(P_1) = 1 - t + t/k$ and $m'(P_i) = t/k$, $i > 1$. Let $\mu = m' \times m$. Define a new partition $Q = \{Q_i\}_{i=1}^{k}$ by $Q_i = \bigcup_{j=1}^{k} P_j \times P_{i+j-1(\mathrm{mod}\, k)}$. Let ζ be the unique probability measure on $\mathcal{E}$ defined by

$$\zeta\left(\bigcap_{\mathbf{v}\in R(m)} S_{\mathbf{v}}(P_j \times P_r)\right) = \mu\left(\bigcap_{\mathbf{v}\in R(m)} S_{\mathbf{v}}(Q_j \cap (X \times P_r))\right)$$

$$= \mu\left(\bigcap_{\mathbf{v}\in R(m)} S_{\mathbf{v}}(P_{r-j+1(\mathrm{mod}\, k)} \times P_r)\right)$$

for all $m \geqq 0$, where j and r may vary with $\mathbf{v}$. Notice that ζ is ergodic, since it is a factor of μ, which is ergodic. Furthermore, $\zeta^2 = m$ and ζ is $\mathcal{S}$-invariant. Several straightforward computations will verify the remainder of the theorem.

LEMMA 3. *Let* $(X, \mu), \mathcal{T}$ *be an ergodic* $\mathbf{Z}^n$*-dynamical system and let* P, Q *and* H *be partitions of* X. *Suppose* $h(\mathcal{T}, P \vee H) > h(\mathcal{T}, Q \vee H)$. *Given* $\varepsilon > 0$ *and* $r \geqq 0$, *there is a partition* $\tilde{P}$ *of* X *such that*

(1) $\tilde{P} \vee H \subset \bigvee_{\mathbf{v}\in\mathbf{Z}^n} T_{-\mathbf{v}}(Q \vee H)$,

(2) $\left|\operatorname{dist}\left(\bigvee_{\mathbf{v}\in R(r)} T_{\mathbf{v}}(P \vee Q \vee H)\right) - \operatorname{dist}\left(\bigvee_{\mathbf{v}\in R(r)} T_{\mathbf{v}}(\tilde{P} \vee Q \vee H)\right)\right| < \varepsilon$ *and*

(3) $h\left(Q \vee H \,\middle|\, \bigvee_{\mathbf{v}\in\mathbf{Z}^n} T_{\mathbf{v}}(\tilde{P} \vee H)\right) < \varepsilon$.

PROOF. Choose $\sigma > 0$ so that $h(\mathcal{T}, P \vee H) - h(\mathcal{T}, Q \vee H) > 3\sigma$. Let $A_m \subset X$ be the set of points x such that

(i) $\left| \frac{1}{|R(m)|} \sum_{v \in R(m)} \mathbf{1}_B(T_v x) - \mu(B) \right| < \sigma$ for all atoms $B \in \bigvee_{v \in R(r)} T_v(P \vee H)$,

(ii) there is a collection $\mathscr{E}_Q$ of sets in $\bigvee_{v \in R(m)} T_v(Q \vee H)$ such that $\mu(\mathscr{E}_Q) > 1 - \sigma$, and

$$2^{-[h(\mathcal{T}, Q \vee H)+\sigma](2m+1)^n} < \mu(E_Q) < 2^{-[h(\mathcal{T}, Q \vee H)-\sigma](2m+1)^n} \qquad \text{for all } E_Q \in \mathscr{E}_Q,$$

and

(iii) there is a collection $\mathscr{E}_P$ of sets in $\bigvee_{v \in R(m)} T_v(P \vee H)$ such that $\mu(\mathscr{E}_P) > 1 - \sigma$, and

$$2^{-[h(\mathcal{T}, P \vee H)+\sigma](2m+1)^n} < \mu(E_P) < 2^{-[h(\mathcal{T}, P \vee H)-\sigma](2m+1)^n} \qquad \text{for all } E_P \in \mathscr{E}_P.$$

By the ergodic theorem and the Shannon–McMillan–Breiman theorem [8], choose M so large that $\mu(A_M) > 1 - (\sigma/4)$. Fix this M. Let $A = A_M$.

Let $\# \mathscr{E}_P$ denote the number of sets in $\mathscr{E}_P$. Then $\# \mathscr{E}_P \geqq \# \mathscr{E}_Q$.

Let $\{H_k \mid k \in \mathscr{K}\}$ be the atoms of $\bigvee_{v \in R(M)} T_v(H)$. Let $\{P_i \mid i \in \mathscr{I}\}$ be the atoms of $\bigvee_{v \in R(M)} T_v(P)$. Let $\{Q_j \mid j \in \mathscr{J}\}$ be the atoms of $\bigvee_{v \in R(M)} T_v(Q)$. Note that $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{K}$ are finite indexing sets.

By the Strong Rochlin Lemma [5], there is a set $\tilde{F} \subset A$ such that $\{T_v \tilde{F}\}_{v \in R(M)}$ are disjoint, $\mu(\bigcup_{v \in R(M)} T_v \tilde{F}) > 1 - (\sigma/2)$ and

$$\operatorname{dist}_\mu \left(\bigvee_{v \in R(m)} T_v(Q \vee H)/\tilde{F} \right) = \operatorname{dist}_\mu \left(\bigvee_{v \in R(m)} T_v(Q \vee H) \right).$$

Decrease the number of columns in this Rochlin tower by replacing the base $\tilde{F}$ with the set $F = \bigcup_{E_Q \in \mathscr{E}_Q} E_Q \cap \tilde{F}$. Then we have a new Rochlin tower $D = \bigvee_{v \in R(M)} T_v F$ which satisfies $\mu(\bigvee_{v \in R(M)} T_v F) > 1 - \sigma$, and also

$$\operatorname{dist}_\mu \left(\bigvee_{v \in R(M)} T_v(Q \vee H)/F \right) = \operatorname{dist}_\mu \left(\bigvee_{v \in R(M)} T_v(Q \vee H) \right).$$

Let $\mathscr{E}_F = \{E_Q \mid E_Q \cap F \neq \varnothing\}$. For any E_P in $\mathscr{E}_P$ and any E_Q in $\mathscr{E}_Q$, we have that $\mu(E_P) < 2^{-\sigma(2M+1)^n} \mu(E_Q)$. Thus, each name in $\mathscr{E}_F$ intersects some collection of names in $\mathscr{E}_P$.

We want to assign to each element of $\mathscr{E}_F$ a unique element of $\mathscr{E}_P$ which intersects it. Choose some E_{Q_1} in $\mathscr{E}_F$ and let $\phi(E_{Q_1}) \in \mathscr{E}_P$ be a P-name which satisfies $E_{Q_1} \cap \phi(E_{Q_1}) \neq \varnothing$. Inductively select an E_{Q_i} (not equal to E_{Q_j} for $j < i$), and assign to it some $\phi(E_{Q_i})$ in $\mathscr{E}_P$ so that $\phi(E_{Q_i}) \neq \phi(E_{Q_j})$ for all $j < i$ and $E_{Q_i} \cap \phi(E_{Q_i}) \neq \emptyset$. Suppose such an assignment cannot be made. Then either (a) all of the elements of $\mathscr{E}_F$ have been used, or (b) all remaining $E_Q \in \mathscr{E}_F$, $Q \neq Q_j$, satisfy $E_P \cap E_Q = \emptyset$ for all $E_P \neq \phi(E_{Q_j})$, $j < i$. If (b) holds then each $E_Q \in \mathscr{E}_F$, $Q \neq Q_j$, must be contained in the set $\bigcup_{j=1}^{i-1} \phi(E_{Q_j}) \cup D'$. If more than half of any E_Q were to lie in D', the tower error set, then $\mu(E_Q) < 2\sigma$, which is false. Hence the "assigned elements" of $\mathscr{E}_P$ must cover at least half of each element $E_Q \in \mathscr{E}_F$, $Q \neq Q_J$, so that the assigned elements of $\mathscr{E}_P$ have total mass greater than $(1-\sigma)/2$.

Therefore,

$$\mu\left(\bigcup_{j=1}^{i-1} E_{Q_j}\right) \geqq 2^{\sigma(2M+1)^n}\mu\left(\bigcup_{j=1}^{i-1} \phi(E_{Q_j})\right) \geqq 2^{\sigma(2M+1)^n}(1-\sigma)/2,$$

which is clearly larger than 1 for M chosen sufficiently large. This is a contradiction, so that (2) cannot occur. Hence, there exists an injective function ϕ which maps the sets of $\mathscr{E}_F$ into the sets of $\mathscr{E}_P$, so that $E_Q \cap \phi(E_Q) \neq \emptyset$.

For $E \in \mathscr{E}_F$, write

$$\phi(E) = \bigcap_{\mathbf{v} \in R(M)} T_{\mathbf{v}}(P_{i(\mathbf{v})} \cap H_{k(\mathbf{v})}), \qquad i(\mathbf{v}) \in \mathscr{I}, \quad k(\mathbf{v}) \in \mathscr{K}.$$

For this E and $\mathbf{v} \in R(M)$, label $T_{\mathbf{v}}(E \cap F)$ by $(i(\mathbf{v}), k(\mathbf{v}))$.

We now define a partition $\tilde{P} = \{\tilde{P}_i\}_{i \in \mathscr{I}}$. Attach a color of R = red to points in the tower and W = white to points not in the tower. This R–W process is a small entropy process, depending on σ, which tracks points in the tower. Let $\tilde{P}_i \cap W = \bigcup_{E \in \mathscr{E}_F} \{T_{\mathbf{v}}(E \cap F) \mid i(\mathbf{v}) = i\}$, a partition on the tower D. For $x \notin D$, let $x \in \tilde{P}_1 \cap R$. This defines a partition $\tilde{P} \vee H$ on X which is clearly measurable with respect to $\bigvee_{\mathbf{v} \in R(M)} T_{\mathbf{v}}(Q \vee H)$, so that (1) of the lemma is satisfied.

Because of the Red and White markers, the tower D is measurable with respect to $\bigvee_{\mathbf{v} \in \mathbf{Z}^n} T_{\mathbf{v}}(\tilde{P} \vee H)$. Since the map ϕ is injective, the $Q \vee H$-names partitioning D are also measurable with respect to $\bigvee_{\mathbf{v} \in \mathbf{Z}^n} T_{\mathbf{v}}(\tilde{P} \vee H)$. Furthermore, the partition $\{D^c, Q_j \cap H_K \cap D\}_{j,k}$ is measurable with respect to the $Q \vee H$-partition of the tower, so that $\{D^c, Q_j \cap H_k \cap D\}_{j,k}$ is measurable with respect to $\bigvee_{\mathbf{v} \in \mathbf{Z}^n} T_{\mathbf{v}}(\tilde{P} \vee H)$. Given $\varepsilon > 0$ choose σ so small that

$$h(Q \vee H \mid \{D^c, Q_j \cap H_k \cap D\}_{j,k}) < \varepsilon.$$

This implies

$$h(Q \vee H \mid \textstyle\bigvee_{v \in Z^n} T_v(\tilde{P} \vee H)) < \varepsilon,$$

which proves (3) of the lemma.

Finally, we prove the distribution relation (2).

Let

$$C = \bigcap_{v \in R(r)} T_v(P_{i(v)} \cap Q_{j(v)} \cap H_{k(v)}) \in \bigvee_{v \in R(r)} T_v(P \vee Q \vee H)$$

and let

$$C' = \bigcap_{v \in R(r)} T_v(\tilde{P}_{i(v)} \cap Q_{j(v)} \cap H_{k(v)}) \in \bigvee_{v \in R(r)} T_v(\tilde{P} \vee Q \vee H)$$

for any choice for $i(v), j(v), k(v)$ and $v \in R(r)$. Let $\varepsilon' = \varepsilon(|Q|\,|P|\,|H|)^{-(r+1)}$. We show that $|\mu(C) - \mu(C')| < \varepsilon'$.

Compute the following:

$$\begin{aligned}\mu(D \cap C) &= \sum_{v \in R(M)} \mu(T_v F \cap C) = \int_F \sum_{v \in R(M)} \mu(\{x \mid x \in T_{-v} C\}) d\mu \\ &= \int_F \sum_{v \in R(M)} \mu(\{x \mid x \in T_{-v} C'\}) d\mu = [\mu(C') \pm \sigma](2m+1)^n \mu(F) \\ &= (1 - \sigma/2)[\mu(C') \pm \sigma].\end{aligned}$$

Furthermore, $0 \leqq \mu(C) - \mu(D \cap C) = \mu(D^c \cap C) \leqq \mu(D^c) < \sigma$. Hence $|\mu(C') - \mu(C)| < 3\sigma$. Choose σ so that $3\sigma < \varepsilon'$, to complete the proof of Lemma 3. ∎

We are ready to prove Theorem 1, which will complete the proof of the Relative Isomorphism Theorem.

Proof of Theorem 1. Show that $\mathcal{M}_1$ is a dense G_δ in $\mathcal{M}_0$.

Write $\mathcal{M}_1 = \bigcap_m \mathcal{O}_m$ where each set $\mathcal{O}_m$ is defined by

$$\mathcal{O}_m = \{\mu \in \mathcal{M}_0 \mid h_\mu(P_1 \vee H_1 \times 2 \mid 2 \times \mathcal{F}_2) < 1/m\}.$$

The upper semi-continuity of the entropy function, with respect to the distribution metric, implies that $\mathcal{O}_m$ is an open set in $\mathcal{M}_0$. The difficulty arises in showing that $\mathcal{O}_m$ is dense in $\mathcal{M}_0$. Once this is proven, the Baire Category Theorem implies that $\mathcal{M}_1$ is a dense G_δ in $\mathcal{M}_0$, which completes the theorem.

Let $\mu \in \mathcal{M}_0$. We must find a relative joining in $\mathcal{O}_m$ which is arbitrarily close to μ in the distribution metric.

Let $\delta > 0$. By definition of $\mathcal{M}_0$, we have an ergodic ν in $\mathcal{M}_0$ such that $d(\mu, \nu) < \delta$. We now work with this ergodic ν.

We know that $h(\nu^1 : P_1 \vee H_1) = h(\nu^2 : P_2 \vee H_2)$, and would like to apply Lemma 3, to "copy" the process $(\nu^1, P_1 \vee H_1)$ into $(\nu^2, P_2 \vee H_2)$. In order to do this, we must boost the entropy of $(\nu^1, P_1 \vee H_1)$. We use Lemma 2.

Let ζ be a δ-randomization of $(\nu^1, P_1) = (m_1, P_1)$, so that the Bernoulli shift used in constructing ζ is independent of the process $(\nu^1, H_1) = (m_1, H_1)$. Consider the process $(\zeta^1, P_1 \vee H_1) = (\zeta^1, P_1) \vee (m_1, H_1)$. Since $(\bigcup_{i=1}^{|P_1|} P_{1_i} \times P_{1_i}) \geqq 1 - \delta$, given $\delta_1 > 0$, we may choose δ so small that $d(\zeta^1, \nu^1) < \delta_1$. Furthermore, if $|P_1| = k$ then

$$h(\zeta^1 : \mathcal{T}^1, P_1 \vee H_1) \geqq (1-\delta) h(m_1 : \mathcal{T}^1, P_1 \vee H_1) + \delta[(\log k) + h(m_1 : H_1)].$$

Thus

$$h(\zeta^1 : \mathcal{T}^1, P_1 \vee H_1) \geqq (1-\delta) h(m_1 : \mathcal{T}^1, P_1 \vee H_1) + \delta[(\log k) + h(m_1 : H_1)].$$

Without loss of generality, we assume that $h(m_1 : \mathcal{T}^1, P_1 \vee H_1) < (\log k) + h(m_1 : H_1)$. This simply means that the process $(m_1 : \mathcal{T}^1, P_1)$ is not the full $\mathbf{Z}^n$-shift on k symbols. To treat this case, we would embed this shift into a shift on $k+1$ symbols, in which case the above assumption holds.

Now $h(m_1 : \mathcal{T}^1, P_1 \vee H_1) < (\log k) + h(m : H_1)$, so that the strict inequality

$$h(\zeta^1 : \mathcal{T}^1, P_1 \vee H_1) > h(m_1 : \mathcal{T}^1, P_1 \vee H_1) = h(m_2 : \mathcal{T}^2, P_2 \vee H_2)$$

holds.

Let λ be an ergodic joining of ζ^1 and m_2 over the common factor $(m_1, H_1) \sim (m_2, H_2)$, such that $\lambda^1 = \zeta^1$ and $\lambda^2 = m_2$. Furthermore, choose λ so that $d(\lambda, \nu) < \delta_1$.

Given δ, choose M so large that if α and β are probabilities on Z with

$$\left| \mathrm{dist}_\alpha \left(\bigvee_{\mathbf{v} \in R(M)} U_\mathbf{v}(P_1 \vee H_1) \times (P_2 \vee H_2) \right) - \mathrm{dist}_\beta \left(\bigvee_{\mathbf{v} \in R(M)} U_\mathbf{v}(P_1 \vee H_1) \times (P_2 \vee H_2) \right) \right| < \delta,$$

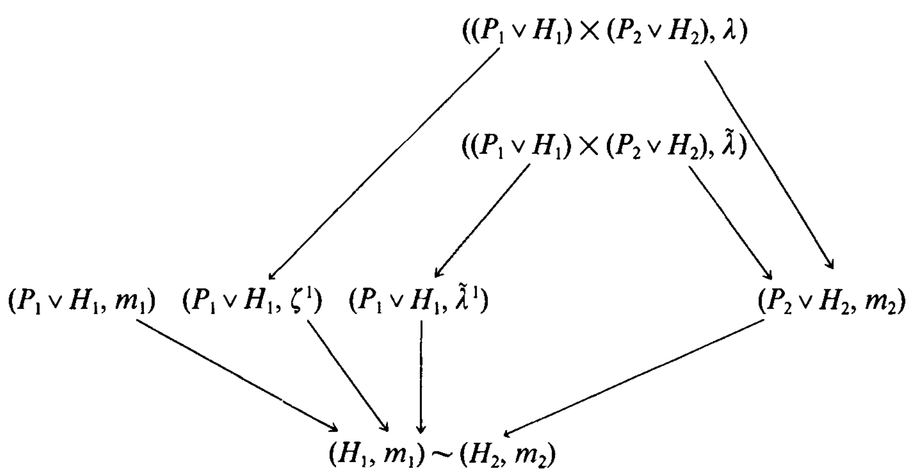
then $d(\alpha, \beta) < 2\delta$.

Apply Lemma 3 to the ergodic system (Z, λ), $\mathcal{U}$ and finite partitions $P_1 \times X_2$, $X_1 \times P_2$ and $H_1 \times X_2 = X_1 \times H_2$. Note that $h_\lambda(\mathcal{U}, (P_1 \vee H_1) \times X_2) > h_\lambda(\mathcal{U}, X_1 \times (P_2 \vee H_2))$. Therefore, by Lemma 2, given δ and M, there is a partition $\tilde{P}_1 \vee H_1 \subset \bigvee_{\mathbf{v} \in \mathbf{Z}^{2n}} U_\mathbf{v}(X_1 \times (P_2 \vee H_2))$ such that

$$\left|\operatorname{dist}_\lambda\left(\bigvee_{v\in R(M)} U_v(P_1 \vee H_1)\times(P_2\vee H_2)\right)\right.$$

$$\left. - \operatorname{dist}_\lambda\left(\bigvee_{v\in R(M)} U_v(\tilde{P}_1 \vee H_1)\times(P_2\vee H_2)\right)\right| < \delta,$$

Transfer λ on $(\tilde{P}\vee H_1)\times(P_2\vee H_2)$-names to $\tilde{\lambda}$ on $(P_1\vee H_1)\times(P_2\vee H_2)$-names so that $\tilde{\lambda}$ is a joining of $((P_1\vee H_1)\times X_2, \zeta^1)$ and $(X_1\times(P_2\vee H_2), m_2)$, relative to the common factor $(H_1, m_1)\sim(H_2, m_2)$, $d(\lambda,\tilde{\lambda})<2\delta$. We have the following diagram:



We will construct an appropriate joining η of $(P_1\vee H_1, m_1)$ with $(P_1\vee H_1, \tilde{\lambda}^1)$ by using the fact that $(P_1\vee H_1, m_1)$ is H_1-rel. f.d. We will let π be the relatively independent joining of η and $\tilde{\lambda}$ over the common factor $(H_1, m_1)\sim(H_2, m_2)$. We will see that for π-a.e. ergodic component π_z, $\pi_z^{1,4}$ will be the desired relative joining of m_1 and m_2 which lies in $\mathcal{O}_m$.

We now construct η. Since $d(\lambda,\tilde{\lambda})<2\delta$,

$$|\operatorname{dist}(\lambda^1 : P_1\vee H_1) - \operatorname{dist}(\tilde{\lambda}^1 : P_1\vee H_1)| < 2\delta.$$

The entropy function is upper semi-continuous with respect to the distribution metric. Thus, given δ_2, we may choose δ so small that $h(\tilde{\lambda}^1 : P_1\vee H_1) - h(\lambda^1 : P_1\vee H_1) < \delta_2$. Furthermore,

$$h(\tilde{\lambda}^1 : P_1 \vee H_1) - h(\lambda^1 : P_1 \vee H_1) = h(\lambda^2 : \tilde{P}_1 \vee H_1) - h(\lambda^1 : P_1 \vee H_1)$$

$$= h(\lambda^2 : \tilde{P}_1 \vee H_1) - h(\lambda^2 : P_2 \vee H_2) + h(\lambda^2 : P_2 \vee H_2) - h(\lambda^1 : P_1 \vee H_1)$$

$$\geqq -h\left(P_2 \vee H_2 \,\middle|\, \bigvee_{v \in \mathbf{Z}^n} T^1_v(\tilde{P}_1 \vee H_1)\right) + h(m_1 : P_1 \vee H_1) - h(\zeta^1 : P_1 \vee H_1)$$

$$> -\delta_2 - \delta_3(\delta_1) \qquad \text{by Lemma 3,}$$

where $\delta_3(\delta_1)$ is some small number depending on δ_1, by upper semi-continuity.

Therefore $|\operatorname{dist}(m_1 : P_1 \vee H_1) - \operatorname{dist}(\tilde{\lambda}^1 : P_1 \vee H_1)| < \delta_1 + 2\delta$ and

$$|h(m_1 : \mathcal{T}^1, P_1 \vee H_1) - h(\tilde{\lambda}^1 : \mathcal{T}^1, P_1 \vee H_1)|$$

$$< \delta_2 + \delta_3(\delta_1) + |h(m_1) - h(\tilde{\lambda}^1)| < \delta_2 + 2\delta_3(\delta_1)$$

by upper semi-continuity.

Thus, since $(m_1, P_1 \vee H_1)$ is H_1-rel. f.d., given any δ_4 we may choose δ, δ_1, δ_2 and δ_3 so small that $\bar{d}_{H_1, H_2}[(m_1, P_1 \vee H_1), (\tilde{\lambda}^1, P_1 \vee H_1)] < \delta_4$. By the definition of relative $\bar{d}$-distance, this gives us a $\mathcal{T}^1 \times \mathcal{T}^1$-invariant ergodic probability measure η on $\mathcal{F}_1 \times \mathcal{F}_1$, a joining of $(m_1, P_1 \vee H_1)$ and $(\tilde{\lambda}^1, P_1 \vee H_1)$, relative to the factor $(m_1, H_1) \sim (m_2, H_2)$, such that $\eta^1 = m_1$, $\eta^2 = \tilde{\lambda}^1$ and $\eta(\bigcup_{i=1}^k P_{1_i} \times P_{1_i}) > 1 - \delta_4$.

Let π be the relatively independent joining of η and $\tilde{\lambda}$ over the common factor $(P_1 \vee H_1, \tilde{\lambda})$. If π is not ergodic, let π_z be an ergodic component of π which is still a joining of η and $\tilde{\lambda}$ over $(P_1 \vee H_1, \tilde{\lambda}^1)$. Let $\rho = \pi_z^{1,4}$ be the projection of π onto its first and fourth marginals. Notice that ρ is ergodic, since it is the marginal of an ergodic measure. Furthermore, $(H_1, m_1) \sim (H_2, m_2)$ is still a factor of ρ. We verify that ρ is close to μ in the distribution metric and that ρ is in $\mathcal{O}_m$.

Let $\varepsilon > 0$. Using the triangle inequality, $d(\mu, \rho) < \delta + \delta_1 + 2\delta + 2\delta_4$. Choose δ, δ_1 and δ_4 so small that $3\delta + \delta_1 + 2\delta_4 < \varepsilon$ to see that $d(\mu, \rho) < \varepsilon$.

By construction, $h_{\tilde{\lambda}}((P_1 \vee H_1) \times X_2 \mid X_1 \times \mathcal{F}_2) = 0$. Since the entropy function is upper semi-continuous, there exists some $\tilde{\delta} = \tilde{\delta}(\tilde{\lambda})$ such that if $d(\tilde{\lambda}, \rho) < \tilde{\delta}$ then $h_\rho((P_1 \vee H_1) \times X_2 \mid X_1 \times \mathcal{F}_2) - h_{\tilde{\lambda}}((P_1 \vee H_1) \times X_2 \mid X_1 \times \mathcal{F}_2) < 1/m$. In particular, $h_\rho((P_1 \vee H_1) \times X_2 \mid X_1 \times \mathcal{F}_2) < 1/m$. Thus, if we further require that $\delta_4 < \tilde{\delta}/2$ then $d(\tilde{\lambda}, \rho) < \tilde{\delta}$ so that $h_\rho((P_1 \vee H_1) \times X_2 \mid X_1 \times \mathcal{F}_2) < 1/m$, which implies $\rho \in \mathcal{O}_m$. ■

Therefore, any two ergodic $\mathbf{Z}^n$-actions which are finitely determined relative to a common factor are isomorphic if and only if they have the same entropy.

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